

TOTAL DESTRUCTION OF INVARIANT TORI FOR THE GENERALIZED FRENKEL-KONTOROVA MODEL

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ABSTRACT. We consider generalized Frenkel-Kontorova models on higher dimensional lattices. We show that the invariant tori which are parameterized by continuous hull functions can be destroyed by small perturbations in the C^r topology with $r < 1$.

Key words. invariant tori, variational methods, hull functions, Percival Lagrangian

AMS subject classifications (2000). 82B20, 37A60, 49J40, 58F30, 58F11

1. INTRODUCTION

The standard Frenkel-Kontorova model:

$$(1) \quad \mathcal{L}(u) = \sum_{i \in \mathbb{Z}} \frac{1}{2}(u_i - u_{i+1})^2 + \frac{\lambda}{2\pi} \cos(2\pi u_i),$$

is the most famous example of models to describe rather crude microscopic theory of plasticity due to dislocations. It also has been considered as a model of deposition of material over a periodic 1-dimensional substratum. The u_i denotes the position of the i th particle and the terms $\frac{1}{2}(u_i - u_{i+1})^2$ and $\frac{\lambda}{2\pi} \cos(2\pi u_i)$ model the energy of interaction of nearest neighboring particles and the energy of interaction with the substratum respectively.

In this paper, we are concerned with the destruction of invariant tori for the generalized Frenkel-Kontorova models on higher dimensional lattices, i.e. the variational problem for “configurations” $u : \mathbb{Z}^d \rightarrow \mathbb{R}$:

$$(2) \quad \mathcal{L}(u) = \sum_{i \in \mathbb{Z}^d} \sum_{j=1}^d H_j(u_i, u_{i+e_j})$$

where $H_j : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $e_j \in \mathbb{R}^d$ is the unit vector in the j th direction. The model (2) is a natural generalization of Frenkel-Kontorova models to more complicated physical lattices in the point of view of the solid state motivation.

We say the Frenkel-Kontorova model admits an invariant torus with rotation vector ω if the configuration u with rotation vector ω can be parameterized by some continuous hull function (see (3) below). The case $d = 1$ corresponds to twist maps. Destruction of invariant tori for this model is closely related to the instability problems in solids, such as diffusions on lattices [Sli99] and so on.

The results in this direction, to the best of our knowledge, is the work of V. Bangert who proved the existence of gaps after large perturbations in the elliptic PDE case [Ban87a]. More precisely, under appropriate non-degenerate conditions on integrand F , [Mos86, Ban87b] proved the existence of Mather set $\mathcal{M}_\alpha^{rec}(F)$. Here $\mathcal{M}_\alpha^{rec}(F)$ is a natural generalization of the set $\mathcal{M}_\alpha^{rec}(F_0) = \{v \mid v(x) = \alpha \cdot x + v_0, v_0 \in \mathbb{R}\}$ for the Dirichlet integrand $F_0(x, v, v_x) = |v_x|^2$. [Ban87a] provided some examples of integrands F such that the graphs of the functions $v \in \mathcal{M}_\alpha^{rec}(F)$ only form a lamination, i.e. a foliation with gaps, for all rotation vectors $\alpha \in \mathbb{R}^n$ with $|\alpha|$ smaller than some arbitrary constant.

From a quite different point of view from V. Bangert's, we will prove the existence of gaps for the generalized Frenkel-Kontorova models after small perturbations by variational methods.

More precisely, we will prove the following theorem:

Theorem 1. *There exists a sequence of C^∞ functions $\{H_j^n\}_{n \in \mathbb{N}}$ converging to the integrable system H_0 with $H_0(x, x') = 1/2(x - x')^2$ in the $C^{1-\epsilon}$ topology such that for n large enough, the generalized Frenkel-Kontorova models produced by H_j^n admit no invariant tori, where $\epsilon > 0$ is a small constant independent of n .*

2. COMPARISON WITH THE LITERATURES

The study of invariant objects on which the dynamics can be conjugated to a rotation is one of the central subjects of dynamical systems and mathematical physics. The celebrated KAM theory ensures the persistence of KAM tori after small perturbation for integrable Hamiltonian systems with non-degeneracy conditions and sufficient differentiability (see [dlL01] and references therein).

The Aubry-Mather theory which had its origin in the work of S. Aubry [ALD83] and J. Mather [Mat82] resembles KAM theory. It produces invariant objects by variational methods which could be Cantor sets in the case of twist maps [Per79, Per80]. For higher dimensional lattices, the corresponding Aubry-Mather theory is recently established in [Bla89, KdlLR97, CdLL98, dLLV07, SdlL11].

The study of non-existence of invariant tori is the complementary to the KAM theory and Aubry-Mather theory. The critical borderline between existence and non-existence of invariant tori in the higher dimensional case is still open and seems to be rather complicated since there are some new phenomena involved.

The problem of destruction of invariant tori under small perturbations has two different flavors as follows:

- a) destruction of invariant torus with a given rotation vector,
- b) total destruction of invariant tori for all rotation vectors.

Case a) is closely related to the arithmetic property of the given rotation vector. Roughly speaking, there is a balance among the arithmetic property of the rotation vector, the regularity of the perturbation and its topology. We mention [Her83, Her86, Mat88, For94] for destruction of invariant circles for twist maps and [Che11] for destruction of KAM torus for Hamiltonian system of multi-degrees of freedom. In particular, [Her83] considered non-existence of invariant circle with

arbitrary given rotation number. [Wan11] provided a variational proof of Herman's result. [Mat88, For94] are concerned with destruction of invariant circles with certain rotation numbers which can be very fast approximated by rational numbers.

Comparing with Case a), due to the absence of arithmetic property, Case b) is much harder and less results are obtained. Naturally, it results in the loss of the regularity of the perturbation to overcome the absence of arithmetic property. A criterion of total destruction plays a crucial role. For twist maps, total destruction of invariant circles is equivalent to the existence of a unbounded orbit. Some criteria for non-existence of invariant tori appeared in [Chi79] and examples of destruction for C^1 small perturbation happened in [Tak71]. Much stronger result was obtained by Herman based on a different geometrical criterion. In [Her83], he proved that for all rotation numbers, invariant circles can be destroyed by C^∞ perturbations close to 0 arbitrarily in $C^{3-\epsilon}$ topology for the generating functions. In [Her0s], Herman extended the above result to the symplectic twist map of multi-degrees of freedom.

Remark 1. *All these subjects have a very long story. For example Appendix A of [CdIL09] contains a mini-survey of the constructive methods to verify the destruction of invariant tori for twist maps.*

3. PRELIMINARIES

In order to use variational methods, it is important to set up the Percival Lagrangian approach to the original variational problem (2) in the following way (see [Per79, Per80, SdIL11] for more details).

We assume that the configurations u are parameterized by a function h —the hull function—and a frequency $\omega \in \mathbb{R}^d$ such that

$$(3) \quad u_i = h(\omega \cdot i).$$

where $i \in \mathbb{Z}^d$ and \cdot is the usual inner product in \mathbb{R}^d .

Heuristically, considering a big box and normalizing the Lagrangian in that big box, when the size of that big box goes to infinity in some sense, we are led to considering

$$(4) \quad \mathcal{P}_\omega(h) = \sum_{j=1}^d \int_0^1 H_j(h(\theta), h(\theta + \omega_j)) d\theta.$$

This imprecise derivation shows that $\mathcal{P}_\omega(h)$ has a direct physical interpretation as the average energy per volume.

Finding the ground states for the variation problem (2) is transformed into finding the minimizers of the variational problem (4). The rigorous study of all these is included in [SdIL11].

We also note that this formalism can be used as the basis of KAM theory to produce smooth solutions under some assumptions such as Diophantine properties of the frequencies, the system is close to integrable, etc. (see [SZ89, CdIL09]).

Assume H_j with the following form:

$$(5) \quad H_j(x, x') = \frac{1}{2}(x - x')^2 + v_j(x).$$

where $v_j(x+1) = v_j(x)$.

We start by summarizing the main concepts of the origin generalized Frenkel-Kontorova model (2) in the sense of calculus of variations.

According to [Mor24], we have

Definition 1. A configuration $u : \mathbb{Z}^d \rightarrow \mathbb{R}$ is called a class-A minimizer for (2) when for every $\varphi : \mathbb{Z}^d \rightarrow \mathbb{R}$ with $\varphi_i = 0$ when $|i| \geq N$, we have

$$(6) \quad \sum_{i \in \mathbb{Z}^d, |i| \leq N+1} \sum_{j=1}^d H_j(u_i, u_{i+e_j}) \leq \sum_{i \in \mathbb{Z}^d, |i| \leq N+1} \sum_{j=1}^d H_j(u_i + \varphi_i, u_{i+e_j} + \varphi_{i+e_j}).$$

The equation (6) can be interpreted heuristically as saying $\mathcal{L}(u) \leq \mathcal{L}(u + \varphi)$ after canceling the terms on both sides that are identical.

Class-A minimizers are also called *ground states* in the mathematical physics literature and *local minimizers* in the calculus of variations literature.

Definition 2. We say that a configuration is a critical point of the action whenever it satisfies the Euler-Lagrange equations:

$$(7) \quad \sum_{j=1}^d \partial_1 H_j(u_i, u_{i+e_j}) + \partial_2 H_j(u_{i-e_j}, u_i) = 0 \quad \text{for every } i \in \mathbb{Z}^d.$$

For every given rotation vector $\omega \in \mathbb{R}^d$, we introduce the corresponding Percival Lagrangian (4) defined on the space of functions:

$$(8) \quad Y = \{ h \mid h \text{ monotone, } h(\theta+1) = h(\theta) + 1, h(\theta_-) = h(\theta) \}$$

where $h(\theta_-)$ (or $h(\theta_+)$) denote the left (or right) limit of h at point θ .

We endow Y with the graph topology defined in the following way.

Denote

$$\text{graph}(h) = \{(\theta, y) \in \mathbb{R}^2 : h(\theta) \leq y \leq h(\theta_+)\}.$$

If $h, \tilde{h} \in Y$ we define the distance between h and \tilde{h} as the Hausdorff distance of their graphs:

$$(9) \quad d(h, \tilde{h}) = \max \left\{ \sup_{\xi \in \text{graph}(h)} \rho(\xi, \text{graph}(\tilde{h})), \sup_{\eta \in \text{graph}(\tilde{h})} \rho(\eta, \text{graph}(h)) \right\}$$

where $\rho(\cdot, \cdot)$ is the Euclidean distance from a point to a set, $\rho(x, S) = \inf_{y \in S} |y - x|$. Note that the graph topology is weaker than the L^∞ topology.

The formal variation of \mathcal{L}_ω yields the following Euler-Lagrange equation for the hull function h :

$$(10) \quad \sum_{j=1}^d [\partial_1 H_j(h(\theta), h(\theta + \omega_j)) + \partial_2 H_j(h(\theta - \omega_j), h(\theta))] = 0.$$

We have the following theorem proved in [SdlL11] (see [Mat82] in the case of twist maps):

Theorem 2. *For any rotation vector $\omega \in \mathbb{R}^d$, there exists a minimizer $h_\omega \in Y$ of \mathcal{P}_ω satisfying (10). Moreover, the configurations defined by (3) are the ground states of (2).*

The description of the ground states are determined by certain properties of h_ω . In particular, we have

Definition 3. *The Frenkel-Kontorova model admits an invariant torus with rotation vector ω if h_ω is a continuous function.*

4. A VARIATIONAL CRITERION OF TOTAL DESTRUCTION OF INVARIANT TORI

In this section, we will give a criterion of existence of invariant tori independent of the rotation vector. First of all, it is easy to prove the following lemma.

Lemma 1. *Let $h_1, h_2 \in Y$, then we have*

$$(11) \quad \left| \int_0^1 [(h_1(\theta + \omega_j) - h_1(\theta))^2 - (h_2(\theta + \omega_j) - h_2(\theta))^2] d\theta \right| \leq 2 \int_0^1 |h_1(\theta) - h_2(\theta)| d\theta.$$

Proof. We set

$$\Delta(\theta) = h_1(\theta + \omega_j) - h_1(\theta) + h_2(\theta + \omega_j) - h_2(\theta),$$

then Δ is periodic with range included in the interval $[0, 2]$. Moreover,

$$(12) \quad \begin{aligned} & \left| \int_0^1 [(h_1(\theta + \omega_j) - h_1(\theta))^2 - (h_2(\theta + \omega_j) - h_2(\theta))^2] d\theta \right| \\ &= \left| \int_0^1 \Delta(\theta) \cdot [(h_1(\theta + \omega_j) - h_2(\theta + \omega_j)) - (h_1(\theta) - h_2(\theta))] d\theta \right| \\ &= \left| \int_0^1 \Delta(\theta) \cdot (h_1(\theta + \omega_j) - h_2(\theta + \omega_j)) - \int_0^1 \Delta(\theta) \cdot (h_1(\theta) - h_2(\theta)) d\theta \right| \\ &= \left| \int_0^1 \Delta(\theta - \omega_j) \cdot (h_1(\theta) - h_2(\theta)) - \int_0^1 \Delta(\theta) \cdot (h_1(\theta) - h_2(\theta)) d\theta \right| \\ &= \left| \int_0^1 (\Delta(\theta - \omega_j) - \Delta(\theta))(h_1(\theta) - h_2(\theta)) d\theta \right| \\ &\leq 2 \int_0^1 |h_1(\theta) - h_2(\theta)| d\theta. \end{aligned}$$

□

From (5) and Lemma 1, we obtain a necessary condition of existence of invariant tori.

Lemma 2. *If the generalized Frenkel-Kontorova models admits an invariant tori, then there exists an continuous $h_1 \in Y$ and a $j \in \{1, \dots, d\}$ such that for every discontinuous $h_2 \in Y$, we have*

$$(13) \quad \int_0^1 [v_j(h_1(\theta)) - v_j(h_2(\theta))] d\theta \leq \int_0^1 |h_1(\theta) - h_2(\theta)| d\theta.$$

Proof. We assume by contradiction that there exists a non-continuous function $h \in Y$ such that for every $j \in \{1, \dots, d\}$

$$(14) \quad \int_0^1 [v_j(h_1(\theta)) - v_j(h_2(\theta))] d\theta > \int_0^1 |h_1(\theta) - h_2(\theta)| d\theta.$$

Due to Lemma 1, we obtain

$$(15) \quad \sum_{j=1}^d \int_0^1 [(h_2(\theta + \omega_j) - h(\theta))^2 - (h_1(\theta + \omega_j) - h_1(\theta))^2] d\theta < 2 \sum_{j=1}^d \int_0^1 [v_j(h_1(\theta)) - v_j(h_2(\theta))] d\theta$$

by summing over the index j . From (4) and (5), we have

$$(16) \quad \mathcal{L}_\omega(h) < \mathcal{L}_\omega(h_\omega).$$

Since h is non-continuous, it is contradicted by the existence of invariant tori. \square

5. PROOF OF THEOREM 1

Based on the preparations above, we will prove Theorem 1 in this section. First of all, we give the construction of H_j^n .

5.1. Construction of the generalized Frenkel-Kontorova models. Consider a sequence of nearly integrable systems generated by

$$(17) \quad H_j^n(x, x') = \frac{1}{2}(x - x')^2 + v_j^n(x)$$

where v_j^n is a non-negative function satisfying

$$(18) \quad \begin{cases} v_j^n(x+1) = v_j^n(x), \\ \max v_j^n(x) = v_j^n(x_0) = \frac{1}{n}, \\ \text{supp } v_j^n \cap [0, 1] = B_R(x_0), \\ R = \left(\frac{1}{n}\right)^{\frac{1}{r}}, \end{cases}$$

where $n \in \mathbb{N}$, $x_0 = \frac{1}{2}$ and r is a positive constant independent of n . From (18), it is easy to see that

$$(19) \quad \|v_j^n\|_{C^{r-\epsilon}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

5.2. Total destruction of invariant tori. From Lemma 2, it suffices to find a discontinuous function $h_2 \in Y$ such that for every continuous function $h_1 \in Y$, we have

$$(20) \quad \int_0^1 [v_j^n(h_1(\theta)) - v_j^n(h_2(\theta))] d\theta > \int_0^1 |h_1(\theta) - h_2(\theta)| d\theta.$$

Without loss of generality, one can assume $h_1(0) = 0$. To achieve this, we construct h_2 in the following way. It is sufficient to define it on $[0, 1)$ due to the fact that $h_2(\theta + 1) = h_2(\theta) + 1$.

We set

$$(21) \quad h_2(\theta) = \begin{cases} h_1(\theta), & 0 \leq \theta \leq A, \\ x_0 - R, & A < \theta \leq B, \\ h_1(\theta), & B < \theta < 1, \end{cases}$$

where $R = n^{-\frac{1}{r}}$ and $A = h_1^{-1}(x_0 - R)$ if $h_1^{-1}(x_0 - R)$ is a single point or $A = \min h_1^{-1}(x_0 - R)$ otherwise and $B = h_1^{-1}(x_0 + R)$ if $h_1^{-1}(x_0 + R)$ is a single point or $B = \max h_1^{-1}(x_0 + R)$ otherwise.

Based on the construction of v_j^n , we have that for $\theta \in [0, 1) \setminus [A, B]$

$$(22) \quad |h_1(\theta) - \theta| \leq \epsilon(n),$$

where $\epsilon(n) \rightarrow 0$ as $n \rightarrow \infty$. Indeed, by (17), for every rotation vector ω ,

$$H_j^n(h_1(\theta), h_1(\theta + \omega)) = \frac{1}{2}(h_1(\theta) - h_1(\theta + \omega))^2 + v_j^n(h_1(\theta)).$$

For $\theta \in [0, 1) \setminus [A, B]$,

$$h_1(\theta) \notin \text{supp } v_j^n,$$

hence, $v_j^n(h_1(\theta)) = 0$. It follows that (22) is verified.

Moreover, a simple calculation implies that

$$(23) \quad \int_0^1 |h_1(\theta) - h_2(\theta)| d\theta \leq C_2 \left(\frac{1}{n} \right)^{\frac{2}{r}}.$$

By (18), we have

$$(24) \quad \int_0^1 v_j^n(h_2(\theta)) d\theta = 0,$$

moreover,

$$(25) \quad \int_0^1 v_j^n(h_1(\theta)) - v_j^n(h_2(\theta)) d\theta = \int_0^1 v_j^n(h_1(\theta)) d\theta = O\left(\left(\frac{1}{n}\right)^{1+\frac{1}{r}}\right).$$

To obtain (20), it is enough to require

$$(26) \quad \frac{2}{r} > 1 + \frac{1}{r}.$$

Hence we get $r < 1$. From (19) and Lemma 2, we complete the proof of Theorem 1.

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